# Generalized Voice-Leading Spaces

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**Supporting Online Material**
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REPORTS

# Generalized Voice-Leading Spaces

**Clifton Callender,[1] Ian Quinn,[2] Dmitri Tymoczko[3]***

**Western musicians traditionally classify pitch sequences by disregarding the effects of five musical transformations: octave shift, permutation, transposition, inversion, and cardinality change. We model this process mathematically, showing that it produces 32 equivalence relations on chords, 243 equivalence relations on chord sequences, and 32 families of geometrical quotient spaces, in which both chords and chord sequences are represented. This model reveals connections between music-theoretical concepts, yields new analytical tools, unifies existing geometrical representations, and suggests a way to understand similarity between chord types.**

To interpret music is to ignore information. A capable musician can understand the sequence of notes ($C_4$, $E_4$, $G_4$) in various ways: as an ordered pitch sequence (for example, an ascending C-major arpeggio starting on middle C), an unordered collection of octave-free note-types (for example, a C major chord), an unordered collection of octave-free note-types modulo transposition (for example, a major chord), and so on. Musicians commonly abstract away from five types of information: the octave in which notes appear, their order, their specific pitch level, whether a sequence appears right-side up or upside down (inverted), and the number of times a note appears. Different purposes require different information; consequently, there is no one optimal degree of abstraction.

Here we model this process. We represent pitches by the logarithms of their fundamental frequencies, setting middle C at 60 and the octave equal to 12. A musical object is a sequence of pitches ordered in time or by instrument (*1*): The object ($C_4$, $E_4$, $G_4$) can represent consecutive pitches played by a single instrument or a simultaneous event in which the first instrument plays $C_4$, the second $E_4$, and the third $G_4$. (Instruments can be ordered arbitrarily.) Musicians generate equivalence classes (*2*, *3*) of objects by ignoring five kinds of transformation: octave shifts (O), which move any note in an object into any other octave; permutations (P), which reorder an object; transpositions (T), which move all the notes in an object in the same direction by the same amount; inversions (I), which turn an object upside down; and cardinality changes (C), which insert duplications into an object (*4*) (fig. S1 and Table 1). (Note that O operations can move just one of an object's notes, whereas T operations move all notes.) We can form equivalence relations with any combination of the OPTIC operations, yielding $2^5 = 32$ possibilities.

A musical progression is an ordered sequence of musical objects. Let $\mathcal{F}$ be a collection of musical transformations, with $f, f_1, \ldots, f_n \in \mathcal{F}$. The progression ($p_1$, …, $p_n$) is uniformly $\mathcal{F}$-equivalent to [ $f(p_1)$, …, $f(p_n)$] and individually $\mathcal{F}$-equivalent to [$f_1(p_1)$, …, $f_n(p_n)$]. Uniform equivalence uses a single operation to transform each object in the first progression into the corresponding object in the second; individual equivalence may apply different operations to a progression's objects (fig. S2). The OPTIC operations can be applied uniformly, individually, or not at all, yielding $3^5 = 243$ equivalence relations on progressions.

A number of traditional music-theoretical concepts can be understood in this way, including chord (OPC), chord type (OPTC), set class (OPTIC), chord-progression (individual OPC), voice leading (uniform OP), pitch class (single notes under O), and many others [table S1 and (*4*)]. We can also combine OPTIC operations in new ways, producing new music-theoretical tools. For example, analogs to voice leadings

**Table 1.** Equivalence relations and quotient spaces produced by the five principal transformations in Western music theory. Here, $\boldsymbol{x}$ is a point in $\mathbb{R}^n$, $\mathbf{1}$ represents (1, …, 1), and $\mathcal{S}_n$ is the symmetric group of order $n$.

| | Equivalence relation | Space |
|---|---|---|
| None | | $\mathbb{R}^n$ |
| Octave | $\boldsymbol{x} \sim_O \boldsymbol{x} + 12\boldsymbol{i}, \boldsymbol{i} \in \mathbb{Z}^n$ | $\mathbb{T}^n$ |
| Transposition | $\boldsymbol{x} \sim_T \boldsymbol{x} + c\mathbf{1}, c \in \mathbb{R}$ | $\mathbb{R}^{n-1}$ or $\mathbb{T}^{n-1}$ (if in conjunction with **O**) (orthogonal projection creates a barycentric coordinate system) |
| Permutation | $\boldsymbol{x} \sim_P \boldsymbol{\sigma}(\boldsymbol{x}), \boldsymbol{\sigma} \in \mathcal{S}_n$ | add $/\mathcal{S}_n$ |
| Inversion | $\boldsymbol{x} \sim_I -\boldsymbol{x}$ | Add $/\mathbb{Z}_2$ [or $/(\mathcal{S}_n \times \mathbb{Z}_2)$ if in conjunction with **P**] |
| Cardinality | $(\ldots, x_i, x_{i+1} \ldots) \sim_C (\ldots, x_i, x_i, x_{i+1} \ldots)$ | Infinite dimensional "Ran space" |

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connect the elements of one chord type (or set class) to those of another; these are OPT (or OPTI) voice-leading classes, resulting from the application of uniform OP (or OPI) and individual T (*1*, *5*) (Fig. 1). These equivalence relations can reveal connections within and across musical works and can simplify the analysis of voice leading by grouping the large number of possibilities into more manageable categories.

Geometrically, a musical object can be represented as a point in $\mathbb{R}^n$. The four OPTI equivalences create quotient spaces by identifying (or "gluing together") points in $\mathbb{R}^n$ (fig. S3). Octave equivalence identifies pitches $p$ and $p$ + 12, transforming $\mathbb{R}^n$ into the $n$-torus $\mathbb{T}^n$. Transpositional equivalence identifies points in $\mathbb{R}^n$ with their (Euclidean) orthogonal projections onto the hyperplane containing chords summing to 0. This transforms $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$, creating a barycentric coordinate system in the quotient (basis vectors pointing from the barycenter of a regular $n$-simplex to its vertices). Permutation equivalence identifies points in $\mathbb{R}^n$ with their reflections in the hyperplanes containing chords with duplicate notes. Musical inversion is represented by geometric inversion through the origin. Permutation and inversion create singular quotient spaces (orbifolds) not locally Euclidean at their fixed points. C equivalence associates points in spaces of different dimension: The result is the infinite-dimensional union of a series of finite subset spaces (*6*–*8*).

One can apply any combination of the OPTI equivalences to $\mathbb{R}^n$, yielding $2^4$ = 16 quotient spaces for each dimension (Table 1); applying C produces 16 additional infinite-dimensional quotients. Any ordered pair of points in any quotient space represents an equivalence class of progressions related individually by the relevant combination of OPTIC equivalences. The image of a line segment in $\mathbb{R}^n$ [a "line segment" in the quotient, although it may "bounce off" a singularity (*1*, *9*)] can be identified with an equivalence class of progressions related uniformly by the relevant combination of OPIC and individually by T. (This is because T acts by orthogonal projection.) Intuitively, pairs of points represent successions between equivalence classes, considered as indivisible harmonic wholes; line segments represent specific connections between their elements.

Music theorists have proposed numerous geometrical models of musical structure (fig. S4 and table S3), many of which are regions of the spaces described in this report. These models have often been incomplete, displaying only a portion of the available chords or chord types and omitting singularities and other nontrivial geometrical and topological features. Furthermore, they have been explored in isolation, without an explanation of how they are derived or how they relate (*10*). Our model resolves these issues by describing the complete family of continuous $n$-note spaces corresponding to the 32 OPTIC equivalence relations.

Of these, the most useful are the OP, OPT, and OPTI spaces, representing voice-leading relations among chords, chord types, and set classes, respectively (*4*). The OP spaces $\mathbb{T}^n/\mathcal{S}_n$ ($n$-tori modulo the symmetric group) have been described previously (*9*). The OPT space $\mathbb{T}^{n-1}/\mathcal{S}_n$ is the quotient of an ($n$ – 1)-simplex, whose boundary is singular, by the rigid transformation cyclically permuting its vertices (*4*). The OPTI space $\mathbb{T}^{n-1}/(\mathcal{S}_n \times \mathbb{Z}_2)$ is the quotient of the resulting space by an additional

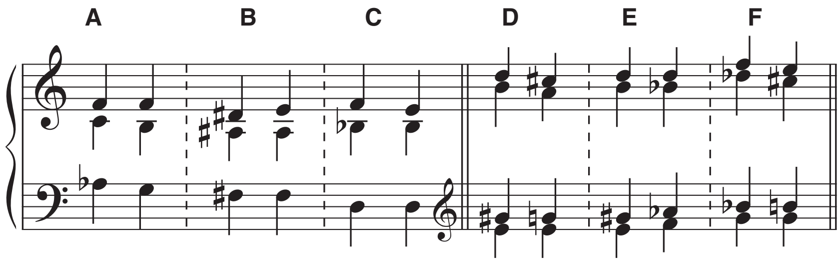



**Fig. 1.** Progressions belonging to the same OPT and OPTI voice-leading classes. Each group exhibits the same underlying voice-leading structure: Analogous elements in the first chord are connected to analogous elements in the second, and the distances moved by the voices are equal up to an additive constant. (**A**) A iv$^6$-V$^7$ progression from Mozart's C minor fantasy, Köchel catalog number (K.) 457, measures 13 and 14. (**B**) A progression from mm. 15-16 of the same piece, individually T-related to (A). (**C**) A progression from Beethoven's Ninth Symphony, movement I, measure 102, related to (A) by individual T and uniform OPI. (**D**) A common voice leading between fifth-related dominant-seventh chords. (**E**) A common voice leading between tritone-related dominant-seventh chords, related to (D) by individual T. (**F**) A voice leading between tritone-related half-diminished sevenths, related to (D) by individual T and uniform I.

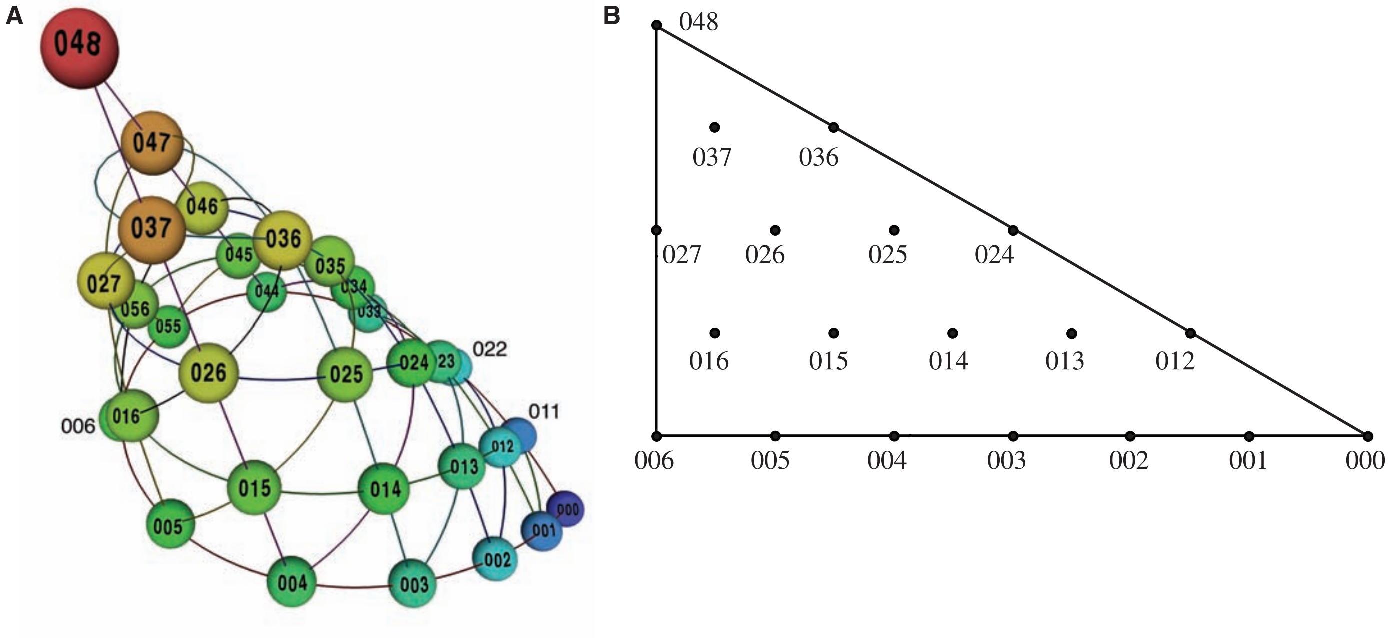



**Fig. 2.** (**A**) $\mathbb{T}^2/\mathcal{S}_3$ is a cone. (**B**) $\mathbb{T}^2/(\mathcal{S}_3 \times \mathbb{Z}_2)$ is a triangle. Numbers refer to pitch classes, with 0 = C, 1 = C♯, etc. Points represent equivalence classes of transpositionally (A) or transpositionally and inversionally (B) related chords. Thus, (C, D, E) and (D, E, F♯) are both instances of 024.

reflection. These spaces can be visualized as oblique cones over quotients of the $(n - 2)$-sphere, albeit with additional orbifold points. The singular point at the cone's "vertex" contains the chord type that divides the octave into $n$ equal pieces; the more closely a chord type's notes cluster together, the farther it is from this point. The singular base of conical OPT space can be visualized as the quotient of the $(n - 2)$-sphere by the cyclic group $\mathbb{Z}_n$. (When $n$ is prime, this is a lens space; when $n$ is not prime, it does not in general seem to have a familiar name.) The base acts like a mirror, containing chords with note duplications. The OPTI-spaces $\mathbb{T}^{n-1}/(\mathcal{S}_n \times \mathbb{Z}_2)$ are essentially similar: They can be visualized as cones over the quotients of the $(n - 2)$-sphere by the dihedral group $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, with $\mathbb{Z}_2$ representing central inversion.

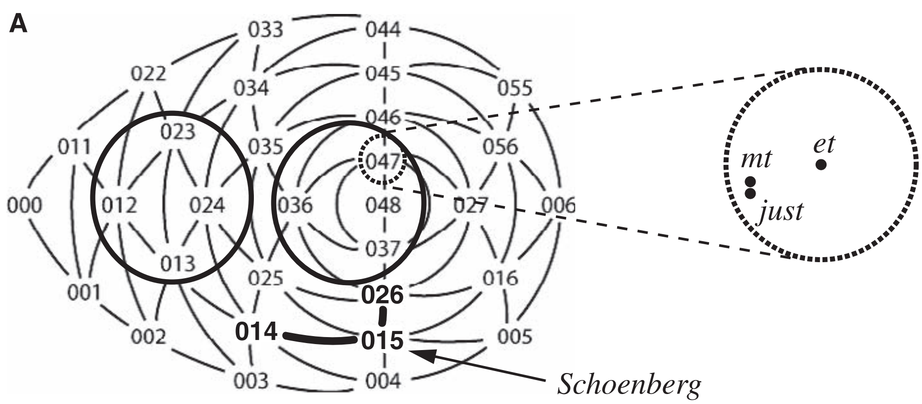
Figure 2A depicts $\mathbb{T}^2/\mathcal{S}_3$, the space of three-note chord types (OPT equivalence classes). The augmented triad at the vertex, 048, divides the octave perfectly evenly; major and minor triads, 047 and 037, are found near the tip and are the basic sonorities of Western tonality. The triple unison 000 occupies the "kink" in the cone's singular base, which acts like a mirror. Orthogonal projection creates a barycentric coordinate system, seen here as a triangular grid (*1*). Pairs of points represent successions of chord types, whereas line segments represent OPT voice-leading classes (Fig. 1). Figure 2B is $\mathbb{T}^2/(\mathcal{S}_3 \times \mathbb{Z}_2)$, the quotient of the cone by a reflection. Points are OPTI equivalence classes (set classes), line segments are OPTI voice-leading classes, and all three boundaries act like mirrors. Figure S5 depicts the analogous four-note structures. Because the spaces are conical, some line segments near the vertex will self-intersect; musically, this means that nearly even chords can be linked to their transpositions by efficient voice leading (*4*).

The advantage of these constructions is that they permit a continuous generalization of traditional music-theoretical terminology. Informal musical discourse recognizes degrees of relatedness: Equal-tempered and just-intonation major triads are considered highly similar, even though they are not related by any OPTIC transformations. Likewise, composers often use scales in which scalar transposition [translation along a scale (*4*)] is nearly equal to log-frequency transposition: In such scales, fragments such as C-D-E ("Do, a deer") and D-E-F ("Re, a drop") are considered similar, even though they are not OPTIC-equivalent. Traditional music theory, however, has often adopted a binary approach to classification: Chords are considered equivalent if they can be related by OPTIC transformations and are considered unrelated otherwise (*2*). Several theorists have recently criticized this view, and modeling similarity between chord types is an active area of music-theoretical research (*4*, *11*). However, no existing model describes the broad flexibility inherent in ordinary musical terminology; for example, none explains the similarity between just and equal-tempered major triads.

Our spaces suggest such a measure. Nearby points represent equivalence classes whose members can be linked by small voice leadings; in this sense, they are nearly equivalent modulo the relevant equivalence relation. This model is in good accord with traditional musical practice. Figure 3 shows that the traditional musical term "scale fragment" refers to a small region of the graph of equal-tempered chords in three-note conical OPT space ($\mathbb{T}^2/\mathcal{S}_3$); likewise, the term "triad" refers to a region of the equal-tempered graph surrounding the vertex. Because the various tunings of the major triad lie very close together in $\mathbb{T}^2/\mathcal{S}_3$, the term "major triad" can be taken to refer to an even smaller region of the continuous space.

Geometrical representations of musical relatedness can also be useful in specific analytical contexts. Figure 3B shows three fragments from an early Schoenberg piano piece, which traditional theory would consider to be unrelated. Some theorists (*12*) have noted that the fragments can be understood as variations on the same musical idea. Following Straus (*13*), we can model Schoenberg's variations geometrically: Fig. 3A shows that the fragments define a sequence of short moves in $\mathbb{T}^2/\mathcal{S}_3$, each producing small changes in the motive's basic intervals. Similar analytical techniques can be used to compare progressions, rather than isolated chords (*4*). Because the OPTIC spaces are continuous, these techniques can potentially be applied to non-Western and microtonal music as well.

Beyond modeling musical similarity, the geometrical perspective provides a unified framework for investigating a wide range of contemporary music-theoretical topics, including "contour" and "K-nets" (*4*, *14*, *15*). This reflects the fact that the OPTIC equivalences have been central to Western musical discourse since at least the seventeenth century (*16*). Our model translates these music-theoretical terms into precise geometrical language, revealing a rich set of mathematical consequences. This translation may have implications for theory, analysis, pedagogy, composition, musical data analysis and visualization, and perhaps even the design of new musical instruments.



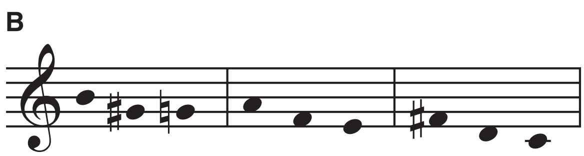




**Fig. 3.** (**A**) The cone $\mathbb{T}^2/\mathcal{S}_3$ (Fig. 2A) as seen from above. The term "scale fragment" refers to graph points in the leftmost circle: "Do, Re, Mi" is 024; "Re, Mi, Fa" is 023; and "Mi, Fa, Sol" is 013. "Triad" refers to points in the larger circle on the right, containing major (047), minor (037), augmented (048), and diminished (036) chords. Different tunings of the major triad are found in the dotted circle around 047. On the right, the equal-tempered, just, and quarter-comma meantone triads. (**B**) Closely related motives at the beginning of Schoenberg's opus 11, no. 1. The variations define a pair of short line segments in the cone, from 014 to 015 to 026.

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